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The String Measure and Spectral Flow of Critical $N = 2$ Strings [1]

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Abstract

The general structure of N=2 moduli space at arbitrary genus *and* instanton number is investigated. The N=2 NSR string measure is calculated, yielding picture- and $U(1)$ ghost number-changing operator insertions. An explicit formula for the spectral flow operator acting on vertex operators is given, and its effect on N=2 string amplitudes is discussed.

1 *Introduction.* The recent interest [1, 2, 3] in computing loop scattering amplitudes of critical $N{=}2$ strings in 2+2 dimensions is partially motivated by a rich structure of $N{=}2$ moduli space, which leads to some novel features of string perturbation theory when compared to the $N{=}1$ and $N{=}0$ cases. In refs. [1, 2, 3], an $N{=}4$ (twisted) supersymmetrical topological description of critical $N{=}2$ strings was used to bypass problems with the (bosonized) $N{=}2$ superconformal ghosts and the location of the $N{=}2$ picture-changing operators. Still, it is of interest to pursue the non-topological approach of integrating correlation functions of BRST-invariant vertex operators over all $N{=}2$ supergravity moduli. We shall show, in particular, how the integration over $U(1)$ moduli generates $U(1)$ ghost number-changing operators and implements the spectral flow for string amplitudes.

In refs. [4, 5, 6], we have investigated the critical $N{=}2$ string in the BRST formalism, but confined ourselves to calculating the BRST cohomology and tree amplitudes without $U(1)$ instantons. In this letter, we are going to describe the structure of the $N{=}2$ moduli space at arbitrary genus *and* $U(1)$ instanton number, and calculate the string measure resulting from integrating out fermionic as well as $U(1)$ moduli.

The gauge-invariant $N{=}2$ string world-sheet action is given by the minimal coupling of $N{=}2$ matter represented by two $N{=}2$ scalar multiplets (X^μ, ψ^μ) [3] to $N{=}2$ supergravity comprising a zweibein e^a_α or a metric $g_{\alpha\beta}$, a real $U(1)$ gauge field A_α, and two real gravitini $\chi^\pm_\alpha$, [4] all living on the two-dimensional (2d) world-sheet. The local symmetries are given by 2d reparametrizations, local Lorentz and Weyl invariances, $N{=}2$ supersymmetry and $N{=}2$ super-Weyl symmetry, phase $U_{\mathrm V}(1)$ and chiral $U_{\mathrm A}(1)$ invariances. [5] The world-sheet Σ is supposed to arise from a closed orientable $N{=}2$ super-Riemann surface, whose topology is characterized by two integers, its Euler number χ (related to the genus h) and its first Chern class (or $U(1)$ instanton

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[3] The fields X^μ and ψ^μ are complex-valued, and $\mu = 0, 1$ is a vector index w.r.t. the spacetime 'Lorentz group' $U(1,1)$.

[4] The superscript $\pm$ of a field denotes its $U(1)$ charge ± 1, and $\alpha, \beta = 0, 1$ are 2d world-sheet indices.

[5] Note that there is only a single gauge field for both $U(1)$ invariances.

number [6]) c,

$$\chi = \frac{1}{2\pi} \int_\Sigma R = 2 - 2h , \quad h \in \mathbf{N} ; \qquad c = \frac{1}{2\pi} \int_\Sigma F = k , \quad k \in \mathbf{Z} . \qquad (1)$$

Here, the curvature two-form R and the $U(1)$ field strength two-form $F = dA$ have been introduced. Finally, correlation functions require the introduction of a third integer, $n \in \mathbf{N}$, which counts the number of punctures on $\Sigma_{h,n}$.

2 *The U(1) bundles.* The $U(1)$ gauge field $A_\alpha = (A_z, A_{\bar{z}})$ is the vertical connection on a principal $U(1)$ bundle over $\Sigma_{h,n}$. It can always be split as $A = A^{\text{inst}} + A^0$, where the instantonic part A^{inst} saturates the instanton number, $c(A^{\text{inst}}) = k$, and A^0 is globally defined on $\Sigma_{h,n}$ since $c(A^0) = 0$. The ambiguity in choosing a representative A^{inst} is contained in the space of A^0's. Since we are going to integrate over the moduli space of all $U(1)$ connections, we need to decompose A^0 into gauge and moduli parts. The Hodge decomposition on $\Sigma_{h,n}$ reads

$$A^0 = d\phi + \delta\omega + A^{\text{Teich}} , \qquad (2)$$

where the function ϕ and two-form ω represent the $U_V(1)$ and $U_A(1)$ gauge degrees of freedom, respectively, and A^{Teich} is a harmonic one-form on $\Sigma_{h,n}$ representing the $U(1)$ Teichmüller degrees of freedom. [7] In the presence of punctures, however, this decomposition is not yet unique. There exist harmonic one-forms which are also exact or co-exact but diverge at the punctures. In locally flat holomorphic coordinates around a puncture, the co-exact prototype is

$$\tilde{A} = \frac{1}{2i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) = d\tilde{\phi} = \delta\tilde{\omega} , \qquad \tilde{\phi} = \frac{1}{2i}\ln\frac{z}{\bar{z}} , \qquad \tilde{\omega} = \frac{i}{2}\ln|z|\, dz \wedge d\bar{z} \quad (3)$$

which is harmonic because

$$\tilde{F} = d\tilde{A} = 2\pi i\, \delta^{(2)}(z)\, dz \wedge d\bar{z} \qquad (4)$$

vanishes away from the puncture, implying $\Delta\tilde{A} = 0$ at least locally on $\Sigma_{h,n}$. $\tilde{A}$ is not exact because $\tilde{\phi}$ is multi-valued around $z{=}0$. Clearly, $\lambda\tilde{A}$ represents a Dirac monopole

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[6] After Wick-rotating the world-sheet to the Euclidean regime, it is better to think of c as a $U(1)$ monopole charge.

[7] We use the notation $\delta = *d*$, where the star means Hodge conjugation, so that $\Delta = \delta d + d\delta$ and $d^2 = 0 = \delta^2$. The connection A^{Teich} is both closed and co-closed, $dA^{\text{Teich}} = 0 = \delta A^{\text{Teich}}$.

with magnetic charge λ. [8] Similarly, one may utilize the Hodge dual of $\tilde{A}$,

$$
\tilde{A} \;=\; -\frac{1}{2}\left(\frac{dz}{z}+\frac{d\bar{z}}{\bar{z}}\right) \;=\; \delta\tilde{\phi} \;=\; -d*\tilde{\omega} \quad, \tag{5}
$$

to manufacture an exact (but not co-exact) harmonic one-form which does not contribute to the curvature F at all. We shall render the decomposition (2) unique by putting such gauge fields into the A^{Teich} part, since they do not really represent $U_{\text{V}}(1)$ or $U_{\text{A}}(1)$ gauge degrees of freedom but do something non-trivial to the punctures. With ϕ and ω being smooth at the punctures, the $U_{\text{A}}(1)$ gauge symmetry changes F but not c. Likewise, the $U(1)$ Teichmüller variations must respect $c(A^0) = 0$,

$$
0 \;=\; \frac{1}{2\pi}\int_{\Sigma_{h,n}} dA^0 \;=\; -\frac{1}{2\pi}\sum_{\ell=1}^{n}\oint_{z_\ell} A^{\text{Teich}} \quad. \tag{6}
$$

A second condition arises from the fact that the Lorentz gauge condition, $\delta A = 0$, can be chosen globally,

$$
0 \;=\; \frac{1}{2\pi}\int_{\Sigma_{h,n}} *\delta A^0 \;=\; -\frac{1}{2\pi}\sum_{\ell=1}^{n}\oint_{z_\ell} *A^{\text{Teich}} \quad. \tag{7}
$$

The complex structure of Σ naturally suggests the complex linear combinations

$$
A^{\pm} \;=\; A^0 \pm i*A^0 \quad, \qquad d^{\pm} \;=\; d \pm i*d \quad, \qquad \phi^{\pm} \;=\; \phi \mp i*\omega \tag{8}
$$

so that

$$
A^{\pm} \;=\; d^{\pm}\phi^{\pm} + A^{\pm}_{\text{Teich}} \quad, \qquad F^{\pm} \;=\; dA^{\pm} \;=\; F^0 \pm i*\delta A^0 \quad, \tag{9}
$$

and the singular one-forms of eqs. (3) and (5) combine to

$$
\tilde{A}^{+} \;=\; \frac{dz}{z} \quad, \qquad \tilde{A}^{-} \;=\; \frac{d\bar{z}}{\bar{z}} \quad. \tag{10}
$$

Independent of c, it is easy to count the dimension of the $U(1)$ Teichmüller space. On $\Sigma_{h,n}$, one has $2h$ real abelian one-forms α_i and β_i, $i = 1,\ldots,h$, which may be chosen to be dual to the basis $\{a_i, b_i\}$ of homology cycles,

$$
\oint_{a_i}\alpha_j = \delta_{ij} \quad, \qquad \oint_{b_i}\alpha_j = 0 \quad, \qquad \oint_{a_i}\beta_j = 0 \quad, \qquad \oint_{b_i}\beta_j = \delta_{ij} \quad, \tag{11}
$$

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[8] Globally, the condition $c(A^0) = 0$ forces the sum of all magnetic charges to vanish.

as well as $2n$ real one-forms of the type given in eqs. (3) and (5), namely two for each puncture. However, eqs. (6) and (7) put two real constraints on the coefficients multiplying the latter, so that the total real dimension equals $2h + 2n - 2$ for $n>0$.

Not all $U(1)$ Teichmüller variations are moduli, however. Whenever the Wilson loops

$$W[A; \gamma] \ = \ \exp\left\{i \oint_\gamma A\right\} \qquad \text{and} \qquad *W[A; \gamma] \ = \ \exp\left\{i \oint_\gamma *A\right\} \qquad (12)$$

become trivial ($= 1$) for some A^{Teich} and all cycles γ, we have encountered a 'big' $U(1)$ gauge transformation. Invariably, this will happen if the coefficients multiplying the harmonic one-forms get large enough, as for $\lambda = 1$ in

$$1 = \exp\left\{i \oint_0 \lambda \tilde{A}\right\} = e^{2\pi i \lambda} \qquad \Longleftrightarrow \qquad i\tilde{A} = g^{-1}dg \quad \text{with} \quad g(z) = e^{i\tilde{\phi}(z)} = \frac{z}{|z|} \, , \tag{13}$$

or for $\varphi = 1$ in

$$1 = \exp\left\{i \oint_{a_1} 2\pi\varphi\alpha_1\right\} = e^{2\pi i \varphi} \qquad \Longleftrightarrow \qquad 2\pi i\alpha_1 = g^{-1}dg \quad \text{with} \quad g(z) = e^{2\pi i \int^z \alpha_1} \, . \tag{14}$$

Therefore, the $U(1)$ moduli space of $\Sigma_{h,n}$ is parametrized by $2h + 2n - 2$ real phases (twists),

$$\oint_{a_i} A = 2\pi\varphi_i \quad , \qquad \oint_{b_i} A = 2\pi\vartheta_i \quad , \qquad \oint_{z_\ell} A = 2\pi\lambda_\ell \quad , \qquad \oint_{z_\ell} *A = 2\pi\mu_\ell \quad , \tag{15}$$

subject to $\sum_\ell \lambda_\ell = 0 = \sum_\ell \mu_\ell$. All phases range from 0 to 1. It is important to note, however, that one may change the value of c simply by shifting the λ_ℓ so that the total phase shift is integral. In summary, one obtains (for $n>0$) a real torus

$$\mathcal{M}_{h,n}^{U(1)} \ = \ \frac{\mathbf{R}^{2(h+n-1)}}{\mathbf{Z}^{2(h+n-1)}} \ . \tag{16}$$

This picture does not depend on the value k of the instanton number c. For a thorough discussion of such matters in the context of $N{=}0$ and $N{=}1$ strings, see ref. [7].

From now on we shall employ the $U_V(1)$ Lorentz gauge, $\delta A^0 = 0$, and its $U_A(1)$ counterpart, $F^0 = 0$, on $\Sigma_{h,n}$, effectively setting $\phi = \omega = 0$ in eq. (2). The $U(1)$ moduli space can then be seen as the moduli space of flat connections on $\Sigma_{h,n}$, which is a product of two factors. One factor is the Jacobian variety of flat $U(1)$ connections on $\Sigma_{h,0}$,

$$J(\Sigma_{h,0}) \ = \ \frac{\mathbf{C}^h}{\mathbf{Z}^h + \Omega\mathbf{Z}^h} \ \sim \ \frac{\mathbf{R}^{2h}}{\mathbf{Z}^{2h}} \ = \ Pic(h, 0) \quad , \tag{17}$$

where Ω is the period matrix of Σ. It is diffeomorphic to the torus $Pic(h,0)$ parametrizing all flat holomorphic line bundles over $\Sigma_{h,0}$, with twists φ_i and ϑ_i of eq. (15) on the homology cycles. The other factor is the torus

$$Pic(0,n) = \frac{\mathbf{R}^{2(n-1)}}{\mathbf{Z}^{2(n-1)}} \tag{18}$$

encoding the $2n-2$ independent twists λ_ℓ and μ_ℓ around the punctures on the Riemann sphere $\Sigma_{0,n}$. The corresponding flat connections are given by linear combinations of the singular one-forms given in eq. (10), for each puncture.

To construct the $N{=}2$ string measure, we need to know the instanton solution A^{inst} and the flat connection A^{Teich} explicitly, in terms of moduli. As far as A^{inst} is concerned, it can be chosen to satisfy the Laplace-Beltrami equation (of motion) and the gauge condition $\delta A^{\text{inst}} = 0$ on the punctureless Riemann surface $\Sigma \equiv \Sigma_{h,0}$. Taken together with the conformal gauge for the 2d metric g, they lead to the simple equations

$$\partial_{\bar{z}} g^{z\bar{z}} \partial_z A_{\bar{z}}^{\text{inst}} = 0 \qquad \text{and} \qquad \partial_z A_{\bar{z}}^{\text{inst}} + \partial_{\bar{z}} A_z^{\text{inst}} = 0 \quad , \tag{19}$$

respectively. Since Σ is compact, orientable and without boundary, eq. (19) implies that $\partial_z A_{\bar{z}}^{\text{inst}} = -\partial_{\bar{z}} A_z^{\text{inst}} \sim g_{z\bar{z}}$. The coefficient is easily fixed by the instanton number constraint, $c(A^{\text{inst}}) = k$,

$$\partial_z A_{\bar{z}}^{\text{inst}} = -\partial_{\bar{z}} A_z^{\text{inst}} = \frac{\pi k}{A} g_{z\bar{z}} \quad , \tag{20}$$

where the total area $A = \int_\Sigma d^2 z \, g_{z\bar{z}}$ of Σ has been introduced. The solution to eq. (20) is (cf. ref. [8])

$$A_z^{\text{inst}}(z,\bar{z}) = -\frac{\pi k}{A} \int_\Sigma d^2 w \, \partial_z K(z,w) g_{w\bar{w}} + \pi k \sum_{i,j=1}^h \omega_i(z) \left| \text{Im}\Omega_{ij} \right|^{-1} \int_{\bar{z}_0}^{\bar{z}} \bar{\omega}_j(\bar{w}) d\bar{w} \quad ,$$

$$A_{\bar{z}}^{\text{inst}}(z,\bar{z}) = +\frac{\pi k}{A} \int_\Sigma d^2 w \, \partial_{\bar{z}} K(z,w) g_{w\bar{w}} - \pi k \sum_{i,j=1}^h \bar{\omega}_i(\bar{z}) \left| \text{Im}\Omega_{ij} \right|^{-1} \int_{z_0}^z \omega_j(w) dw \quad ,$$

$$\tag{21}$$

where $K(z,w)$ is the Green function to the scalar Laplacian,

$$\partial_z \partial_{\bar{z}} K(z,w) = \delta_{z\bar{z}}^{(2)}(z,w) + \omega_i(z) \left| \text{Im}\Omega_{ij} \right|^{-1} \bar{\omega}_j(\bar{z}) \quad , \tag{22}$$

and $\omega_i = \alpha_i + \Omega_{ij}\beta_j$, $i = 1,\ldots,h$, are the holomorphic abelian differentials with normalization

$$\oint_{a_i} \omega_j = \delta_{ij} \quad , \qquad \oint_{b_i} \omega_j = \Omega_{ij} \quad . \tag{23}$$

The solution to eq. (22) can be expressed in terms of the prime form $E(z, w)$ [9] as

$$K(z, w) \;=\; \ln |E(z, w)|^2 + \sum_{i,j=1}^{h} \left[\operatorname{Im} \int_{w}^{z} \omega_i(u) du\right] |\operatorname{Im}\Omega_{ij}|^{-1} \left[\operatorname{Im} \int_{w}^{z} \omega_j(u) du\right] \;. \quad (24)$$

The fields $A_{z,\bar{z}}^{\text{inst}}$ are neither holomorphic nor single-valued around the homology cycles a_i or b_j but change by a gauge transformation (as long as k is integer).

Turning to A^{Teich}, a convenient parametrization is given by

$$A^{\text{Teich}} \;=\; 2\pi \sum_{i=1}^{h} (\varphi_i \alpha_i + \vartheta_i \beta_i) + \sum_{\ell=1}^{n} (\lambda_\ell^+ \tilde{A}_\ell^+ + \lambda_\ell^- \tilde{A}_\ell^-) \quad, \quad (25)$$

where, in locally flat complex coordinates,

$$A_\ell^+(z) \;=\; \frac{dz}{z - z_\ell} \qquad \text{and} \qquad A_\ell^-(z) \;=\; \frac{d\bar{z}}{\bar{z} - \bar{z}_\ell} \;. \quad (26)$$

3 *The gravitini bundles.* The 2d gravitini $\chi_\alpha^\pm$ transform inhomogeneously under the $N{=}2$ local supersymmetry and $N{=}2$ super-Weyl (fermionic) gauge symmetry as

$$\delta_{\text{S}} \chi_\alpha^\pm \;=\; \hat{D}_\alpha \varepsilon^\pm \quad, \qquad \delta_{\text{W}} \chi_\alpha^\pm \;=\; \gamma_\alpha \zeta^\pm \quad, \quad (27)$$

where $\hat{D}_\alpha(\hat{\omega}, A)$ is the $N{=}2$ supergravitational covariant derivative containing the spin connection $\hat{\omega}_\alpha$ and the $U(1)$ gauge connection A_α. The $\chi_\alpha^\pm$ transform homogeneously under all other local symmetries, with a $U(1)$ charge of ± 1. Hence, the gravitini are sections of some complex spinor bundle associated to the principal $U(1)$ bundle over $\Sigma_{h,n}$.

The local symmetries of eq. (27) allow one to gauge away all 8 real Grassmann degrees of freedom of $\chi_\alpha^\pm$, except for those in the kernel of $\hat{D}^\pm$,

$$\hat{D}^\pm \chi_z^\pm \;\equiv\; \left(\bar{\partial} \mp i A_{\bar{z}}\right) \chi_z^\pm \;=\; 0 \quad, \quad (28)$$

and similarly for $\chi_{\bar{z}}^\pm$ (signs are correlated). We have used the superconformal gauge, in which $\hat{\omega}_{\bar{z}} = 0$ and all χ's are γ-traceless. The solutions of eq. (28) are the fermionic moduli on $\Sigma_{h,n}$. Their number depends on h and n as well as on the instanton number k and is dictated by the Riemann-Roch theorem,

$$\operatorname{ind}\hat{D}^\pm \;\equiv\; \dim \ker \hat{D}^\pm - \dim \ker \hat{D}^{\pm\dagger} \;=\; 2(h-1) \pm k + n \quad. \quad (29)$$

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[9] The prime form $E(z, w)$ is a holomorphic $(-\frac{1}{2}, 0)$ form in z and w, with a single zero at $z = w$.

For $h > 1$, the contributions to a positive index generically come from the first term, so that on reads off $2(h-1)+n+k$ positively charged and $2(h-1)+n-k$ negatively charged fermionic moduli. When the index becomes negative, $\det \hat{D}^{\pm^\dagger}$ develops zero modes, which implies the vanishing of the corresponding n-point correlator. This restricts the range of the sum over k to

$$|k| \;\leq\; 2(h-1)+n \quad . \tag{30}$$

The issue of a complex structure for the gravitini bundles with $k \neq 0$ is a subtle one since $A_{\bar{z}}$ in $\hat{D}^{\pm}$ contains the non-holomorphic $A_{\bar{z}}^{\text{inst}}$ of eq. (21). Even without punctures ($n=0$), the gravitini bundles cannot in general be *holomorphic* line bundles. The latter are (twisted) integral or half-integral powers of the canonical line bundle and as such always yield integral multiples of $h-1$ for $\text{ind}\hat{D}$. The r.h.s. of eq. (29), however, is of this form only when k itself is a multiple of $h-1$, in which case the gauge connection may be absorbed in the spin connection, effectively shifting the conformal weights of the gravitini from $\frac{3}{2}$ to $\frac{3}{2}$, to $\frac{3}{2}\pm\frac{1}{2}$, or to $\frac{3}{2}\pm1$.

4 *The string measure.* Our starting point is the formal expression for scattering amplitudes,

$$A_n \;=\; \sum_{h,k}\frac{1}{\mathcal{N}}\int D(X\psi g\chi A)\,e^{-S_{\text{m}}}\,V_1\ldots V_n \quad , \tag{31}$$

where S_{m} is the gauge-invariant $N{=}2$ string (matter) action, V_ℓ represent vertex operators for particles, and $\mathcal{N}$ denotes the volume of the gauge group. We fix the $N{=}2$ superconformal gauge and use the BRST method [5]. A careful treatment of the Faddeev-Popov determinant, naively [10]

$$\int D(bc\beta\gamma\tilde{b}\tilde{c})\,e^{-S_{\text{gh}}} \quad , \tag{32}$$

yields anti-ghost zero mode insertions for each moduli direction. As we already know from $N{=}0$ and $N{=}1$ string theory [7], these anti-ghost insertions come paired with the corresponding Beltrami differentials which are the tangents to the moduli slice. Let us take $h>1$ for simplicity; the cases of the sphere and torus require obvious minor

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[10] Our notation is as follows (see refs. [4, 5] for more details): (b, c) stand for conformal ghosts, $(\beta^{\pm}, \gamma^{\pm})$ for $N{=}2$ superconformal ghosts, and $(\tilde{b}, \tilde{c})$ for $U(1)$ ghosts. The ghosts for the Weyl and super-Weyl symmetries are ignored since they do not propagate. S_{gh} is the ghost action.

modifications due to isometries. We get

$$\left| \prod_{m=1}^{3(h-1)+n} \int_\Sigma \mu_m b \right|^2 = \left| \prod_{m=1}^{3(h-1)+n} \oint_{C_m} b \right|^2 \tag{33}$$

for the metric moduli, where the Beltrami differentials $(\mu_m)^z_{\bar{z}} = \nabla_{\bar{z}}(v_m)^z$ have been represented in terms of quasiconformal vector fields v_m with a unit jump across closed contours C_m. The commuting superconformal ghosts yield

$$\left| \prod_{a^+=1}^{2(h-1)+k+n} \delta\big(\beta^+(z_{a^+})\big) \prod_{a^-=1}^{2(h-1)-k+n} \delta\big(\beta^-(z_{a^-})\big) \right|^2 \tag{34}$$

for the $N{=}2$ fermionic moduli, with a delta-function choice for the fermionic Beltrami differentials. [11] Finally, as a novel feature we obtain

$$\prod_{i=1}^{h} \left[\oint_{a_i} \tilde{b} \oint_{b_i} \tilde{b} \right] \left| \prod_{\ell=1}^{n-1} \oint_{z_\ell} \tilde{b} \right|^2 \tag{35}$$

for the $U(1)$ moduli, after taking the real abelian one-forms for the $U(1)$ Beltrami differentials. It must be added that this counting presumes that the vertex operators V_ℓ are taken from the natural picture- and ghost-number sector, namely $(\pi^+, \pi^-) = (-1, -1)$ and of $\tilde{c}c$ type. [12]

The $N{=}2$ supergravity fields enter the full (BRST-invariant) $N{=}2$ string action $S_{\text{tot}} = S_{\text{m}} + S_{\text{gh}}$ as Lagrange multipliers [4, 5]. Since the action S_{tot} is *linear* in the fermionic and $U(1)$ moduli, we may integrate those out and arrive at an additional insertion of

$$\left| \prod_{a^+=1}^{2(h-1)+k+n} G^+_{\text{tot}}(z_{a^+}) \prod_{a^-=1}^{2(h-1)-k+n} G^-_{\text{tot}}(z_{a^-}) \prod_{\ell=1}^{n-1} \delta\big(\oint_{z_\ell} J_{\text{tot}}\big) \right|^2 \prod_{i=1}^{h} \left[\delta\big(\oint_{a_i} J_{\text{tot}}\big) \delta\big(\oint_{b_i} J_{\text{tot}}\big) \right] \tag{36}$$

where $G^\pm_{\text{tot}}$ and J_{tot} are the full (BRST-invariant) supercurrents and $U(1)$ current of the $N{=}2$ string, respectively [4, 5].

Combining eqs. (33)–(36), we find a product of $N{=}2$ picture-changing operators (*cf.* ref. [9]),

$$\left| \prod_{m=1}^{3(h-1)+n} \oint_{C_m} b \prod_{a^+=1}^{2(h-1)+k+n} Z^+(z_{a^+}) \prod_{a^-=1}^{2(h-1)-k+n} Z^-(z_{a^-}) \prod_{\ell=1}^{n-1} Z^0(z_\ell) \right|^2 \prod_{i=1}^{h} \left[Z^0(a_i)\, Z^0(b_i) \right] \tag{37}$$

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[11] Like in the $N{=}1$ string, this raises issues of globality and boundary terms in moduli space. We have no comment here.

[12] This allows only NS states; R states will be generated below.

where

$$Z^{\pm} := \delta(\beta^{\pm}) G_{\text{tot}}^{\pm} = \{Q_{\text{BRST}}, \xi^{\pm}\} \tag{38}$$

and

$$Z^0(\gamma) := \left(\oint_{\gamma} \tilde{b}\right) \delta\left(\oint_{\gamma} J_{\text{tot}}\right) \tag{39}$$

for a closed contour γ (being a homology cycle or encircling a puncture). Up to n of the b insertions can be used to convert c-type vertex operators to integrated ones, and any number of $Z^{\pm}$ may be taken to upwardly change their $(-1, -1)$ pictures, keeping the total picture at $(2(h-1)+k, 2(h-1)-k)$. The novel $U(1)$ picture-changing operators Z^0 serve two purposes: First, a Z^0 associated with a homology cycle enforces a projection onto charge-neutral excitations propagating across the loop and ensures factorization on neutral states when pinching the cycle. Second, a Z^0 attached to a puncture can be absorbed by the $\tilde{c}$-type vertex operator, changing its $U(1)$ ghost number by removing the $\tilde{c}$ factor. In this way, only a single $\tilde{c}$-type vertex operator remains in the end, plus the $2h$ real Z^0 insertions associated with the homology cycles. Thus not only $Z^{\pm}$ but also Z^0 provides a map between BRST cohomology classes. [13]

Eq. (37) is formally BRST-invariant, which is important for the consistency and BRST-invariance of $N=2$ string amplitudes. The integration over the $N=2$ matter fields does not present a principal problem, but it has to be done in the presence of a background instanton field A^{inst} minimally coupled to the fermions. No sum over fermionic spin structures appears since it has already been carried out as part of the $U(1)$ moduli integration. The integration over the metric moduli follows the lines of the familiar $N=0$ and $N=1$ cases.

5 *The spectral flow.* The $N=2$ NSR string fermionic matter fields $\psi^{\pm\mu}$ are sections of a complex twisted spinor bundle, just like the gravitini. For vanishing instanton number, $k=0$, this becomes a twisted holomorphic spinor bundle. The associated spin structures parametrizing the NS/R sectors or monodromies of ψ in an n-point function are labeled by the half-points $(\frac{1}{2}\mathbf{Z}/\mathbf{Z})^{2(h+n-1)}$ in the Picard variety of eq. (16). Now observe that by a unitary transformation via

$$U(z) = \exp\left\{i \int_{z_0}^{z} \hat{A}\right\} \tag{40}$$

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[13] Neither $Z^{\pm}$ nor Z^0 have a local inverse [5], while Z^0 is nilpotent.

of a given spinor bundle we can always change the monodromies with a suitable $\hat{A} \in \{A^{\text{Teich}}\}$, because the ψ carry $U(1)$ charge [10]. We can in fact reach any point in the Picard variety and, in particular, move to any other spin structure. Since the unitary transformation of eq. (40) is equivalent to a shift in the integration variables $A^{\text{Teich}} \rightarrow A^{\text{Teich}} + \hat{A}$ and the $U(1)$ moduli space has no boundary, we conclude that the sum over the NSR spin structures (and, in fact, over all intermediate monodromies) is automatically contained in the integration over the $U(1)$ moduli. This is the so-called *spectral flow* of Ooguri and Vafa [11].

As a consequence, the distinction between NS and R sectors is $U(1)$ moduli-dependent and thus cannot be physical. This feature is not restricted to $h \geq 1$, but appears just as well for the n-punctured sphere, i.e. for tree-level $N{=}2$ string amplitudes. More precisely, any pair of R-type punctures can be turned into a pair of NS-type, since there are $n{-}1$ independent cycles and the sum of all twists has to vanish. Hence, the R-type and NS-type states of the $N{=}2$ string cannot be physically distinguished and their correlators must coincide, which is consistent with our previous explicit calculations [5]. Note, however, that the correlation functions should depend on the value k of c, which may be changed by allowing for a non-zero but integral total twist of the external states.

A shift in the $U(1)$ puncture moduli, $\lambda_\ell^+ \rightarrow \lambda_\ell^+ + \theta_\ell$, which shifts the puncture monodromies by θ_ℓ, also modifies the vertex operators $V_\ell(z_\ell)$ present in the string path integral,

$$V_\ell(z_\ell) \longrightarrow V_\ell^{\theta_\ell}(z_\ell) \equiv SFO(\theta_\ell, z_\ell)\, V_\ell(z_\ell) \quad , \tag{41}$$

where, remarkably, the spectral-flow operator $SFO(\theta, z)$ can be written in the explicit form [14]

$$SFO(\theta, z) = \exp\left\{\theta \int^z J_{\text{tot}}(z')dz'\right\} = \exp\{\theta\, (\phi^+ - \phi^- - \varphi^+ + \varphi^- + \tilde{b}c)(z)\} \; . \tag{42}$$

Here, we have bosonized [5]

$$J_{\text{tot}} = \{Q_{\text{BRST}}, \tilde{b}\} = -\frac{1}{2}\psi^+ \cdot \psi^- + \partial(\tilde{b}c) + \frac{1}{2}(\beta^+\gamma^- - \beta^-\gamma^+) \tag{43}$$

in the holomorphic basis via ($\epsilon = \pm 1$)

$$\psi^{\pm,\epsilon} = 2e^{\epsilon\phi^{\pm\epsilon}} \quad , \qquad \gamma^\pm = \eta^\pm e^{+\varphi^\pm} \quad , \qquad \beta^\pm = e^{-\varphi^\mp}\partial\xi^\pm \quad , \tag{44}$$

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[14] Normal ordering is implicit in our formulae. $SFO(\theta)$ generalizes the instanton number-changing operators $SFO(\pm 1)$ of refs. [12, 13].

and obtained a *local* operator. It is easy to check that *SFO* is BRST invariant but not BRST trivial. However, like for the $Z^\pm$, the derivative ∂SFO is BRST trivial, so we may move spectral-flow operators around at will on the world-sheet. Therefore, an insertion of $\prod_\ell SFO(\theta_\ell, z_\ell)$ is BRST-equivalent to unity as long as $\sum_\ell \theta_\ell = 0$, implying again the invariance of string amplitudes under shifts of the puncture monodromies, i.e.

$$\left\langle V_1 V_2 \ldots V_n \right\rangle = \left\langle V_1^{\theta_1} V_2^{\theta_2} \ldots V_n^{\theta_n} \right\rangle \qquad \text{for} \qquad \sum_\ell \theta_\ell = 0 \quad . \tag{45}$$

The $N{=}2$ superconformal algebra generated by $(T_{\text{tot}}, G_{\text{tot}}^\pm, J_{\text{tot}})$ is extended to the *small* $N{=}4$ superconformal algebra by adding the $SU(2)$ ladder operators $SFO(\pm 1)$ and closing the algebra.

The NS$\leftrightarrow$R exchange is accomplished at, $\theta = \pm\frac{1}{2}$, which may be symbolically written as $SFO^\pm \text{NS} = \text{R}^\pm$ where $SFO^\pm \equiv SFO(\pm\frac{1}{2})$ and $\text{NS(R)} \equiv V_{\text{NS(R)}}$. The index on R indicates that we can flow to two different Ramond vertex operators. The $N{=}2$ string vertex operators are known to exist in different (holomorphic) pictures (π^+, π^-), which are connected by the process of picture-changing $(\pi^+, \pi^-) \to (\pi^+{+}1, \pi^-)$ or $(\pi^+, \pi^-) \to (\pi^+, \pi^-{+}1)$ via the picture-changing operators Z^+ or Z^- of eq. (38), respectively. The spectral flow operators $SFO^\pm$ can actually be interpreted as yet additional picture-changing operators:

$$SFO^\pm : \qquad (\pi^+, \pi^-) \to (\pi^+{\pm}\tfrac{1}{2}, \pi^-{\mp}\tfrac{1}{2}) \quad . \tag{46}$$

This leads, for example, to the following identifications among tree-level two-, three- and four-point functions at instanton number $k{=}0$:

$$\left\langle \text{NS NS} \right\rangle = \left\langle \text{R}^+ \text{R}^- \right\rangle \quad , \quad \left\langle \text{NS NS NS} \right\rangle = \left\langle \text{NS R}^+ \text{R}^- \right\rangle \quad ,$$
$$\left\langle \text{NS NS NS NS} \right\rangle = \left\langle \text{NS NS R}^+ \text{R}^- \right\rangle = \left\langle \text{R}^+ \text{R}^- \text{R}^+ \text{R}^- \right\rangle = 0 \quad , \tag{47}$$

which were all verified by explicit calculations in ref. [5]. At tree-level, non-vanishing correlators require $|k| \leq n{-}2$, so that the complete three-point amplitude, for example, also has $k{=}\pm 1$ contributions. These can be generated from $k{=}0$ by inserting $SFO(\pm 1) = SFO^\pm SFO^\pm$ into $\langle \text{NS NS NS} \rangle$, resulting in

$$\left\langle \text{NS R}^+ \text{R}^+ \right\rangle + \left\langle \text{NS R}^- \text{R}^- \right\rangle \quad . \tag{48}$$

Noting that $V_{\text{R}}^- = f(p) V_{\text{R}}^+$ with a momentum-dependent factor $f(p)$ [5], we relate eq. (48) to the $k{=}0$ case and find that the two terms in eq. (48) cancel each other

on-shell. This leaves us with the standard $U(1,1)$-invariant result for the tree-level three-point function and removes the $U(1,1)$ non-invariant $k{\neq}0$ terms (see also ref. [14]). It would be interesting to check whether such a mechanism ensures global $U(1,1)$ symmetry in general.

Finally, it should be noticed that the discussion of spectral flow heavily relied on the use of the *holomorphic* basis for bosonization. In contrast, in the *real* basis (discussed also at length in refs. [5, 14]) the spectral flow is obscured and the operator defined by eq. (43) is non-local.

References

[1] N. Berkovits and C. Vafa, Nucl. Phys. **B433** (1995) 123.

[2] N. Berkovits, *Vanishing Theorems for the Self-Dual N=2 String*,
San Paulo preprint IFUSP–P–1134, December 1994, hep-th/9412179.

[3] H. Ooguri, *Loop Amplitudes of N=2 Strings*, Talk given at the Intern. Conference "Strings '95", Univ. of Southern California, Los Angeles, 13–18 March 1995, to appear in the Proceedings (World Scientific).

[4] S. Ketov, O. Lechtenfeld and A. Parkes, *Twisting the N=2 String*,
DESY and Hannover preprint, DESY 93–191, ITP–UH–24/93, December 1993, hep-th/9312150; to appear in Phys. Rev. **D** (1995).

[5] J. Bischoff, S. Ketov and O. Lechtenfeld, Nucl. Phys. **B438** (1995) 373.

[6] O. Lechtenfeld, *From N=2 Fermionic Strings to Superstrings?*,
Talk given at the 28th Intern. Symposium on the Theory of Elementary Particles, Wendisch-Rietz, Germany, 30 August – 03 September 1994, hep-th/9412242.

[7] E. D'Hoker and D. Phong, Rev. Mod. Phys. **60** (1988) 1012.

[8] F. Ferrari, Helv. Phys. Acta **67** (1994) 702.

[9] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93;
E. Martinec, Phys. Lett. **171B** (1986) 189;
E. Verlinde and H. Verlinde, Nucl. Phys. **B288** (1987) 357; Phys. Lett. **192B** (1987) 95.

[10] L. Alvarez-Gaumé, G. Moore and C. Vafa, Commun. Math. Phys. **106** (1986) 1.

[11] H. Ooguri and C. Vafa, Mod. Phys. Lett. **A5** (1990) 1389; Nucl. Phys. **B361** (1991) 469.

[12] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A9** (1994) 653.

[13] N. Berkovits, *Super-Poincaré Invariant Superstrings*,
San Paulo preprint IFUSP–P–1143, March 1995, hep-th/9503099.

[14] H. Lü and C. N. Pope, *BRST Quantization of the N=2 String*,
Texas and Trieste preprint, CTP–TAMU–62/94, SISSA–175/94/EP,
November 1994, hep-th/9411101.